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Infinite symmetry group in D-dimensional conformal quantum field theory

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Abstract. The diffeomorphism group of the D-dimensional space as the symmetry group for conformal quantum field theory models is proposed. The problem of calculation of critical exponents turns out to be reduced to the study of special diffeomorphism group representations. The general properties of these representations on the space of symmetric traceless tensors are investigated.

1. Introduction

It is well known, that a symmetry in a model of the quantum field theory can essentially simplify its study. The symmetry puts restrictions on the possible form of the Green functions and enables it to make classifications of the field operators. The more transformations the symmetry group contains, the more there are restrictions and the more precise the classifications are. Many very important results in the two-dimensional quantum field theory were obtained by using the infinite group of the symmetry of the two-dimensional local conformal transformations only [1]. The group of the D-dimensional conformal transformations is finite by $D \neq 2$. Therefore, the direct use of it is less effective.

In this paper, we want to show how the infinite group of the general coordinate transformations in the D-dimensional space can be used as the symmetry group for models of the conformal quantum field theory. This group is not the symmetry group of a Lagrangian. It is similar to the symmetry group of the Virasoro algebra. The fields operator are not invariant with respect to its action, but its representation is realized on the field operator set. The use of the symmetry of such a kind in the D-dimensional conformal quantum field theory was suggested in [2].

We shall consider the theory of the scalar field $\phi(x)$ with the action $S(\phi)$ in the D-dimensional Euclidean space, and denote $\Phi(x) = \Phi(\phi(x))$ the local composite field operators. In general, they are tensors with respect to the rotation transformations. The rotation and translation symmetries of the theory are assumed. Therefore, the

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non-trivial tensor structure of the field operator can be obtained by employing the derivative operations with respect to coordinates of fields only. The Green functions of the fields $\Phi(x)$ are of the form:

$$\langle\Phi_1(x_1)\ldots\Phi_n(x_n)\rangle \equiv \int \mathrm{e}^{-S(\phi)}\,\Phi_1(\phi(x_1))\ldots\Phi_n(\phi(x_n))\,\mathrm{D}\phi. \tag{1}$$

As the common as the functional integrals are invariant under the change of the integration variable of the form

$$\int F(\phi)\,\mathrm{D}\phi = \int F(\phi')\,\mathrm{D}\phi'. \tag{2}$$

It is assumed here that the function set over which one makes integration, is the same in both sides of the equality. Making the infinitesimal transformation $\phi \to \phi' = \phi + \delta\phi$ in the right-hand side of (1) we obtain in virtue of (2) the relation:

$$\langle \delta S \Phi_1 \ldots \Phi_n\rangle = \sum_{i=1}^{n} \langle \Phi_1 \ldots \delta\Phi_i \ldots \Phi_n\rangle \tag{3}$$

where

$$\delta S \equiv \frac{\delta S}{\delta\phi}\,\delta\phi \qquad \delta\Phi_i \equiv \frac{\delta\Phi_i}{\delta\phi}\,\delta\phi.$$

If $\delta\phi$ is the field variation of the general form, (3) is equivalent to the Schwinger equation:

$$\left\langle \frac{\delta S}{\delta\phi}\Phi_1 \ldots \Phi_n\right\rangle = \sum_{i=1}^{n}\left\langle \Phi_1 \ldots \frac{\delta\Phi_i}{\delta\phi}\ldots\Phi_n\right\rangle.$$

The Schwinger equations with the initial conditions define all the Green functions of the field theoretical model uniquely, hence using the invariance of the integral with respect to the general field transformation one obtains the complete information of the system. If $\delta\phi$ is the field variation of a special form, the equations (3) contain the part of the information only. If this field transformation is one of some symmetry group, the equalities of the type (3) are called Ward identities. They are the restrictions which the symmetry puts on the Green functions.

2. General coordinate transformations

Let $x \to x'(x)$ be the coordinate transformation of the general form, and $\phi \to \phi'$ is the special field transformation of the form:

$$\phi(x) \to \phi'(x) = \det\left\{\frac{\partial x'}{\partial x}\right\}^{\Delta/D} \phi(x'). \tag{4}$$

Here Δ is a constant which is called the field dimension, D is the dimension of the space. It is easy to see that (4) is the representation of the general coordinate transformation (diffeomorphism) group. In the infinitesimal form

$$x \to x' = x + \alpha(x)$$

$$\phi(x) \to \phi'(x) = \phi(x) + \delta_\alpha\phi(x),\ \delta_\alpha\phi(x) \equiv \left(\alpha(x)\partial + \frac{\Delta}{D}\partial\alpha(x)\right)\phi(x).$$

Let us denote $L_\alpha = (\alpha\partial + (\Delta/D)\partial\alpha)$. Then we have

$$\delta_\alpha\delta_\beta\phi(x) = \delta_\alpha L_\beta\phi(x) = L_\beta L_\alpha\phi(x). \tag{5}$$

It follows directly from the definition of L_α that:

$$L_\alpha L_\beta - L_\beta L_\alpha \equiv [L_\alpha, L_\beta] = L_{[\alpha,\beta]} \tag{6}$$

where $[\alpha(x), \beta(x)]_\mu \equiv (\alpha(x)\partial)\beta_\mu(x) - (\beta(x)\partial)\alpha_\mu(x)$ is the commutator of the vector fields in the D-dimensional space. It has the common properties of the Lee brackets. It follows from (5), (6) that

$$[\delta_\alpha, \delta_\beta] = \delta_{[\beta,\alpha]} \tag{7}$$

i.e. the operators $\delta'_\alpha \equiv -\delta_\alpha$ form the algebra which coincide with the algebra of the diffeomorphism. We suppose by the definition for the classical fields $\Phi(\phi)$ that

$$\delta_\alpha\Phi \equiv \frac{\delta\Phi}{\delta\phi}\delta_\alpha\phi.$$

Then the common Leibnitz rule

$$\delta_\alpha(\Phi_1(x)\Phi_2(y)) = \delta_\alpha\Phi_1(x)\Phi_2(y) + \Phi_1(x)\delta_\alpha\Phi_2(y) \tag{8}$$

is fulfilled for δ_α and the commutation relation (7), too. Substituting in (3) $\delta_\alpha\Phi$ in place of $\delta\Phi$ and denoting $J_\alpha \equiv \delta_\alpha S$ we obtain the Ward identity of the form:

$$\langle J_\alpha\Phi_1\ldots\Phi_n\rangle = \sum_{i=1}^{n}\langle\Phi_1\ldots\delta_\alpha\Phi_i\ldots\Phi_n\rangle. \tag{9}$$

The Group of the conformal transformations is the subgroup of the diffeomorphism group. For the infinitesimal conformal transformation $x \to x + \alpha$,

$$\alpha_\mu(x) = a_\mu + \omega_{\mu\nu}x_\nu + \lambda x_\mu + (x^2\delta_{\mu\nu} - 2x_\mu x_\nu)b_\nu \tag{10}$$

where a_μ, $\omega_{\mu\nu} = -\omega_{\nu\mu}$, λ, b_ν are constant. Summation over repeated indices will be understood throughout. In virtue of the translation and rotation invariance $J_\alpha = 0$ by $\alpha_\mu = a_\mu$ or $\alpha_\mu = \omega_{\mu\nu}x_\nu$. Therefore:

$$J_\alpha = \int dx\,\partial_\mu\alpha_\nu(x)T_{\mu\nu}(x) \qquad T_{\mu\nu} = T_{\nu\mu}.$$

The tensor $T_{\mu\nu}$ is the energy-momentum tensor. If there is the dilatation invariance, $J_\alpha = 0$ by $\alpha_\mu = \lambda x_\mu$, and $T_{\mu\nu}$ is traceless. It can be easily proved in this case that $J_\alpha = 0$ by $\alpha = x^2b - 2(xb)x$, i.e. the conformal invariance follows from the scale invariance. We obtain then from (9):

$$\sum_{i=1}^{n}\langle\Phi_1\ldots\delta_\alpha\Phi_i\ldots\Phi_n\rangle = 0$$

if α is of the form (10).

3. Basic assumptions

The set $F^{(n)}$ of all the n-rank tensor fields is the linear vector space. Let $\Phi^i_{\mu_1,\ldots,\mu_n}$, $i = 1, 2, \ldots$, are its basis elements. They are supposed to be chosen in the form of the product of field derivatives and Kronecker symbols. We assume that δ_α is the linear operation on the set $F=\bigcup_n F^{(n)}$ of all the fields which action on the basis elements is of the following form:

$$\delta_\alpha \Phi^k_{\mu_1,\ldots,\mu_n} = \sum_{j=1}^{\infty} \left(L^{(k,j)}_{\mu_1,\ldots,\mu_n}(\alpha)\Phi^j + \sum_{m=1}^{\infty} L^{(k,j)\nu_1,\ldots,\nu_m}_{\mu_1,\ldots,\mu_n}(\alpha)\Phi^j_{\nu_1,\ldots,\nu_m} \right) \tag{11}$$

where L is a local differential operator. We suppose that the field operators Φ form the containing a unit element algebra:

$$\Phi_i(x)\Phi_j(y) = \int K^l_{ij}(x, y, z)\Phi_l(z)\,\mathrm{d}z. \tag{12}$$

The commutation relations (7) and Leibnitz rule (8) for $x \neq y$ are supposed, too. As it can be easily seen all these assumptions are right for the classical fields. The equalities like (12) are commonly assumed in the quantum field theory and called the Wilson operator product expansions [3]. The general properties of Wilson expansions in conformal invariant quantum field theory were investigated in many papers [4].

In virtue of (11) it is naturally to decompose the set of all the fields into field families. We suppose by definition that each field belongs to one family only and two fields belong to the same family if they can be connected by relation (11). Making the symmetrization and the antisymmetrization over the indexes of the basis tensor fields of the family and subtracting traces, one obtains the irreducible basis. Then, the family of fields can be decomposed into the set of subfamilies. We propose that there are *subfamilies of symmetric traceless tensors and consider in this paper, such families only.*

Let us introduce for convenience of the future scalar functions of the form

$$\Phi^i(u, x) = \Phi^i(x) + \sum_{k=1}^{\infty} \frac{1}{k!}\Phi^i_{\mu_1,\ldots,\mu_k}(x)u_{\mu_1}\ldots u_{\mu_k} \equiv \sum_k \frac{1}{k!}\Phi^i_{\mu_1,\ldots,\mu_k}u_{\mu_1}\ldots u_{\mu_k}$$

for such a family. For the family under consideration, we introduce the characteristic function

$$\chi^i(x) = \sum_{n=0}^{\infty} \frac{a^i_n}{n!} x^n$$

where $a^i_n = 1$, if there is the tensor Φ^i of the rank n in the family, otherwise $a^i_n = 0$.

It will be convenient for us to use the argument contracting operation (ACO). Applied to the function of n arguments, ACO makes in the result the $n-2$ argument function. If two arguments of a function are denoted with the same letters and there is the line over one of these, it means ACO applied to this function. By definition,

$$f(\ldots, z, \ldots, \bar{z}, \ldots) \equiv \mathrm{e}^{\partial^2/\partial x \partial y} f(\ldots, x, \ldots, y, \ldots)|_{x=y=0}.$$

Using these notations, we can rewrite the relation (11) in the more simple form:

$$\delta_\alpha \Phi(u) = L_\alpha(u, v)\Phi(\bar{v}). \tag{13}$$

Here, the number of the fields and their arguments are omitted. If not misleading, we shall use in future the shorthand notations of such a kind, too. By definition,

$$L^{(ij)}_\alpha(u, v; x) \equiv \sum_k \sum_l \frac{1}{k!} L^{(ij)\nu_1,\ldots,\nu_l}_{\mu_1,\ldots,\mu_k}(\alpha(x))u_{\mu_1}\ldots u_{\mu_k}v_{\nu_1}\ldots u_{\nu_l}.$$

It is obvious, that

$$\delta_\alpha\delta_\beta\varphi(u) = L_\beta(u, v)L_\alpha(\bar{v}, w)\Phi(\bar{w}).$$

Hence, in virtue of (6)

$$L_\alpha(u,w)L_\beta(\bar{w},v)-L_\beta(u,w)L_\alpha(\bar{w},v)=L_{[\alpha,\beta]}(u,v). \tag{14}$$

Omitting in (14) the arguments of the functions, we obtain the commutation relation (6). Thus, the representation of the D-dimensional diffeomorphism algebra is realized on the operators L_α if their product is understood as the ACO-product.

The function $H^{ij}(u,v)=\chi^i(uv)\delta_{ij}$ is obviously the projector:

$$H(u,\bar{w})H(w,v)=H(u,v)$$

i.e. $H^2=H$, and

$$H\Phi=\Phi \qquad L_\alpha=HL_\alpha=L_\alpha H.$$

It will be called the characteristic operator.

Let

$$L_\alpha'^{(ij)}(u,v)\equiv e^{uw}\left(\frac{\Delta_i}{D}\partial\alpha+\alpha\partial+(u\partial)(\alpha w)\right)H^{ij}(\bar{w},v). \tag{15}$$

If there is the conformal invariance in the model of quantum field theory, the natural restriction arises for the basis tensor fields. They must be the tensors of the conformal group. It means that the operator L_α must coincide with L'_α, for α having the form (10), i.e. for such α,

$$\delta_\alpha\Phi=L'_\alpha\Phi.$$

Since all the basis tensors are traceless,

$$\frac{\partial^2}{\partial u^2}\Phi(u)=0.$$

Hence,

$$\Phi(u)=P(u,\bar{v})\Phi(v). \tag{16}$$

Here, $P(u,v)$ is the projector, which satisfies the following equations:

$$\frac{\partial^2}{\partial u^2}P(u,v)=\frac{\partial^2}{\partial v^2}P(u,v)=0,\ P(u,w)P(\bar{w},v)=P(u,v).$$

Their solution has the form: $P(u,v)=F(uv,u^2v^2)$, and for $x^2\leq y$

$$F(x,y)=\frac{\Gamma(\rho)}{2\pi i}\oint\frac{(4z)^\rho\, e^{-z}\, dz}{(y-4zx+4z^2)^\rho} \qquad \rho\equiv\frac{D-2}{2}.$$

Here, the integrand function is defined on the complex z-plane with three horizontal cuts, going from the points $z_1=0$, $z_2=(x+\sqrt{x^2-y})$ and $z_3=(x-\sqrt{x^2-y})$ to $-\infty$. The integration contour encircles all the cuts in the positive direction.

4. Equations for Green functions and dimensions of fields

If the operator L_α is known, then one can obtain in the quantum conformal field theory linear equations for Green functions and transcendental ones for the field dimensions. In order to show how it can be done, we consider one of the field families and denote

$$G^i(u, u', x-x') \equiv \langle \Phi^i(u, x)\Phi^i(u', x')\rangle$$

the generating function for the propagators of the fields $\Phi^i_{\mu_1\dots\mu_n}(x)$.

Let $\tilde{G}^i(u, u', x-x')$ be the generating function for the inverse propagators. Then

$$\int G^i(u, w, x-z)\tilde{G}^i(\bar{w}, v, z-y)\,\mathrm{d}z = \delta(x-y)\chi^i(uv).$$

In the conformal quantum field theory the propagators are defined by the field dimensions up to constant factors [5, 6]. For example, the propagator $G(x-y)$ of the scalar field is of the form

$$G(x-y) = c(x-y)^{-2\Delta}$$

where c is constant and Δ is the dimension of the field. The basis of fields in the family can be chosen in such a way that

$$\langle \Phi^i(u, x)\Phi^j(u', x')\rangle = 0 \qquad \text{for } i \neq j.$$

It is convenient to introduce the "shadow" field $\tilde{\Phi}$:

$$\tilde{\Phi}(u, x) \equiv \int \tilde{G}(u, w, x-y)\Phi(\bar{w}, y)\,\mathrm{d}y.$$

Obviously,

$$\langle \Phi^i(u, x)\tilde{\Phi}^j(u', x')\rangle = \delta(x-x')H^{ij}(uu'). \tag{17}$$

It is easy to see that the transformation law of the shadow field has the form:

$$\delta_\alpha\tilde{\Phi} = \tilde{G}\delta_\alpha\Phi = \tilde{L}_\alpha\tilde{\Phi} \qquad \tilde{L}_\alpha \equiv \tilde{G}L_\alpha G.$$

Hence, it follows from the Ward identity and (17) that the function D_α in the Wilson expansion

$$J_\alpha\Phi(u, x) = \int D_\alpha(u, v, x, y)\Phi(\bar{v}, y)\,\mathrm{d}y \tag{18}$$

is of the form:

$$D^{ij}_\alpha(u, v, x, y) = L^{ij}_\alpha(u, v, x)\delta(x-y) + \tilde{L}^{ji}_\alpha(v, u, y, x) \tag{19}$$

or in short-hand notations:

$$D_\alpha = L_\alpha + \tilde{L}^T_\alpha.$$

In virtue of (13), the Ward identity (9) can be written in the form:

$$\langle J_\alpha\Phi_1(u_1, x_1)\dots\Phi_n(u_n, x_n)\rangle$$

$$= \sum_{k=1}^{n} \langle \Phi_1(u_1, x_1)\dots L_\alpha(u_k, v; x_k)\Phi_k(\bar{v}, x_k)\dots\Phi_n(u_n, x_n)\rangle. \tag{20}$$

Now, it is easy to obtain from (18), (19) and (20) the following equation for the Green functions:

$$\langle \tilde{L}^T_\alpha\Phi\Phi_1\dots\Phi_n\rangle = \sum_{k=0}^{n} \langle \Phi\Phi_1\dots L_\alpha\Phi_k\dots\Phi_n\rangle. \tag{21}$$

This equation contains the parameter $\alpha(x)$. Hence, it is equivalent to the infinite number of linear equations for Green functions. The fields Φ, $i=1,\dots,n$ in (21) can belong to different families. In this case, the operator L_α in the right-hand side of (21) is meant to be one of the corresponding family.

In the conformal quantum field theory, the 3-point Green function $G(x_1,x_2,x_3)=\langle\Phi_1(x_1)\Phi_2(x_2)\Phi_3(x_3)\rangle$ is known up to finite numbers of constant factors [5, 6]. For example, if Φ_i, $i=1,2,3$, are scalar fields and the dimensions of them are $\Delta_1,\Delta_2,\Delta_3$, the Green function G is of the form:

$$G(x_1,x_2,x_3)=\frac{c}{(x_1-x_2)^{\Delta'-\Delta_3}(x_2-x_3)^{\Delta'-\Delta_1}(x_3-x_1)^{\Delta'-\Delta_2}}. \tag{22}$$

Here, $\Delta'=\Delta_1+\Delta_2+\Delta_3$ and c is constant. The function G of tensor fields is a sum of functions similar to (22) [6]. The function G must satisfy the equation of the form (21). It is possible for certain field dimensions only. Substituting G in the corresponding equation of (21), one obtains the transcendental equation for the field dimensions. A similar method based on equations of a kind (21) was suggested in [7] for calculations of field dimensions in the D-dimensional conformal quantum field theory.

5. General properties of operator L_α

In virtue of the locality L_α, this differential operator can be written in the form

$$L_\alpha(u,v,x)=i\int e^{ipx}\alpha_\mu(p)l_\mu(u,v,p,z)\,dp \qquad z\equiv -i\frac{\partial}{\partial x}. \tag{23}$$

Substituting (23) in (14), we obtain the equation for the function $l_\mu(u,v,p,z)$:

$$l_\mu(p,z+q)l_\nu(q,z)-l_\nu(q,z+p)l_\mu(p,z)=q_\mu l_\nu(p+q,z)-p_\nu l_\mu(p+q,z). \tag{24}$$

Here, the first two arguments of the functions l are omitted. The product of them in the left-hand side is understood as the ACO-product. For example,

$$l_\mu(p,z+q)l_\nu(q,z)\equiv l_\mu(u,w,p,z+q)l_\nu(\bar w,v,q,z).$$

In virtue of (16), L_α can be written in the form:

$$l_\mu(u,v,p,z)\equiv P(u,w)m_\mu(\bar w,s,p,z)P(\bar s,v). \tag{25}$$

It is assumed that the function $m_\mu(u,v,p,z)$ is independent from u^2, v^2. Since for $\alpha_\mu(x)$ of the form (10) L_α in (13) coincides in the conformal invariant theory with L'_α (15), the following relations must be fulfilled for the function m_μ:

$$m_\mu(0,z)=z_\mu H \qquad \text{(translation: } \alpha_\mu(x)=a_\mu) \tag{26}$$

$$\frac{\partial m_\mu}{\partial p_\nu}(u,v,o,z)-\frac{\partial m_\nu}{\partial p_\mu}(u,v,o,z)=e^{uw}(u_\nu w_\mu-u_\mu w_\nu)H(\bar w,v) \tag{27}$$

$$\text{(rotation: } \alpha_\mu(x)=\omega_{\mu\nu}x_\nu)$$

$$P\frac{\partial m_\mu}{\partial p_\mu}P(u,v,o,z)=e^{uw}(\Delta+uw)H(\bar w,v) \qquad \text{(dilatation: } \alpha_\mu(x)=\lambda x_\mu) \tag{28}$$

$$P\frac{\partial^2 m_\mu}{\partial p_\nu\partial p_\nu}P(u,v,o,z)=2P\frac{\partial^2 m_\nu}{\partial p_\mu\partial p_\nu}P(u,v,o,z) \tag{29}$$

$$\text{(special conformal transformation: } \alpha_\mu(x)=(x^2\delta_{\mu\nu}-2x_\mu x_\nu)b_\nu).$$

In (26)-(28), H is the characteristic operator, Δ is the matrix: $\Delta^{ij}=\Delta_i\delta_{ij}$. By using (25), the commutation relation (24) can be rewritten in the following form:

$$Pm_\mu(p,z+q)Pm_\nu(q,z)P-Pm_\nu(q,z+p)Pm_\mu(p,z)P$$
$$=q_\mu Pm_\nu(p+q,z)P-p_\nu Pm_\mu(p+q,z)P. \tag{30}$$

Then, the equations:

$$Hm_\mu=m_\mu H=m_\mu$$
$$(D_{\mu\nu}(z)+D_{\mu\nu}(p)+D_{\mu\nu}(u)+D_{\mu\nu}(v))m_\lambda(u,v,p,z)$$
$$=\delta_{\nu\lambda}m_\mu(u,v,p,z)-\delta_{\mu\lambda}m_\nu(u,v,p,z)$$
$$\left(p_\nu\frac{\partial}{\partial p_\nu}+z_\nu\frac{\partial}{\partial z_\nu}+u_\nu\frac{\partial}{\partial v_\nu}-u_\nu\frac{\partial}{\partial u_\nu}\right)m_\mu(u,v,p,z)$$
$$=\Delta m_\mu(u,v,p,z)-m_\mu(u,v,p,z)\Delta+m_\mu(u,v,p,z) \tag{31}$$
$$P(u,\bar{w})\left\{\left[(D_{\mu\lambda}(s)+D_{\mu\lambda}(z))\frac{\partial}{\partial z_\lambda}+D_{\mu\lambda}(p)\frac{\partial}{\partial p_\lambda}-s_\lambda\frac{\partial}{\partial s_\lambda}\frac{\partial}{\partial z_\mu}\right]m_\nu(w,s,p,z)\right.$$
$$\left.-\frac{\partial}{\partial z_\mu}m_\nu(w,s,p,z)\Delta\right\}P(\bar{s},v)$$
$$=P(u,\bar{w})\left[\frac{\partial}{\partial p_\nu}m_\mu(w,s,p,z)-\frac{\partial}{\partial p_\mu}m_\nu(w,s,p,z)-\delta_{\mu\nu}\frac{\partial}{\partial p_\lambda}m_\lambda(w,s,p,z)\right.$$
$$\left.+\frac{1}{2}\left(z_\mu\frac{\partial^2}{\partial z^2}+p_\mu\frac{\partial^2}{\partial p^2}\right)m_\nu(w,s,p,z)\right]P(\bar{s},v)$$

where

$$D_{\mu\nu}(p)\equiv p_\mu\frac{\partial}{\partial p_\nu}-p_\nu\frac{\partial}{\partial p_\mu}$$

follow from (30) and equalities (26)-(29).

Let us denote $x_0=uv$, $x_1=up$, $x_2=uz$, $x_3=vp$, $x_4=vz$, $x_5=p^2$, $x_6=pz$, $x_7=z^2$. Then, the function m_μ can be written in the form:

$$m_\mu(u,v,p,z)=a(X)p_\mu+b(X)z_\mu+c(X)v_\mu+d(X)u_\mu. \tag{32}$$

Here, $a(X)$, $b(X)$, $c(X)$, $d(X)$ are some analytical functions of eight arguments $x_0,\ldots,x_7$. Obviously,

$$\left(p\frac{\partial}{\partial p}+z\frac{\partial}{\partial z}+v\frac{\partial}{\partial v}-u\frac{\partial}{\partial u}\right)x_i=y_i$$

where $y=0$ for $i=0,1,2$ and $y_i=2x_i$ for $i=3,4,\ldots,7$. Hence, it follows from (31), (32) that

$$\sum_{i=3}^{7}x_i\frac{\partial}{\partial x_i}a^{ij}=\frac{\Delta_i-\Delta_j}{2}a^{ij}\qquad\sum_{i=3}^{7}x_i\frac{\partial}{\partial x_i}b^{ij}=\frac{\Delta_i-\Delta_j}{2}b^{ij}$$
$$\sum_{i=3}^{7}x_i\frac{\partial}{\partial x_i}c^{ij}=\frac{\Delta_i-\Delta_j}{2}c^{ij}\qquad\sum_{i=3}^{7}x_i\frac{\partial}{\partial x_i}d^{ij}=\frac{\Delta_i-\Delta_j+2}{2}d^{ij}.$$

Since a, b, c, d are analytical, one can conclude from these relations that $\Delta_i - \Delta_j = 2k$, where k is zero or integer. Hence, a, b, c, d are polynoms in $x_3, x_4, \ldots, x_7$ of power k (functions a, b, c) or $k+1$ (function d). Coefficients of these polynoms are arbitrary functions of x_0, x_1, x_2. The powers of them are non-negative. Therefore, $a^{ij} = b^{ij} = c^{ij} = 0$, if $\Delta_i - \Delta_j < 0$ and $d^{ij} = 0$, if $\Delta_i - \Delta_j < -2$.

6. Simple example

The simplest family, on which the non-trivial representation of the diffeomorphism group can be realized, is the set Φ^1, Φ^2, $\Phi^1_{\mu\nu}$, containing two scalar fields and one tensor of the second rank. One can easily find for such a case the solutions m_μ, of commutation relations (30), having the form (32). There are two solutions of the following form:

$$a^{11} = \frac{1}{D}[\Delta(1 + \tfrac{1}{2}x_0^2 + \beta x_1^2) - 2\beta x_1 x_2] \qquad b^{11} = 1 + \tfrac{1}{2}x_0^2 + \beta x_1^2$$

$$c^{11} = x_0 x_1,\ d^{11} = \frac{2\beta x_1 x_7}{D - 2 - 2\Delta} \qquad a^{22} = \frac{\Delta + 2}{D}$$

$$b^{22} = 1 \qquad c^{22} = d^{22} = a^{12} = b^{12} = c^{12} = c^{12} = 0 \qquad d^{12} = \gamma x_1 \tag{33}$$

$$a^{21} = \tau\{[(D-2)x_6 + \Delta x_5](\Delta + 1) - D x_7\}$$

$$b^{21} = \tau D[(\Delta + 1)x_5 + D x_6] \qquad c^{21} = \frac{2x_3}{D\gamma} \qquad d^{21} = 0$$

and

$$a^{11} = \frac{1}{D}\left[\Delta + (\Delta + 4)\frac{x_0^2}{2}\right] \qquad b^{11} = 1 + \frac{x_0^2}{2} \qquad d^{11} = -x_0 x_3$$

$$a^{22} = \frac{\Delta + 2}{D} \qquad b^{22} = 1 \qquad c^{21} = \frac{2x_3}{D} \tag{34}$$

$$a^{12} = a^{21} = b^{12} = b^{21} = c^{11} = c^{12} = c^{22} = d^{12} = d^{21} = d^{22} = 0$$

where Δ, β, γ are arbitrary constants,

$$\tau \equiv \frac{4\beta}{D^2(2 + 2\Delta - D)\gamma}.$$

For the second solution (34), the tensor field $\Phi_{\mu\nu}$ is a contravariant tensor with respect to diffeomorphism transformations. The field operators of such a kind do not exist in scalar field theory, where tensors are obtained by using of derivative operations and Kronecker symbols. The operator L_α corresponding to (33) generates the following transformations of fields:

$$\delta_\alpha \Phi^1 = \frac{\Delta}{D}\partial\alpha\Phi^1 + \alpha\partial\Phi^1$$

$$\delta_\alpha \Phi^2 = \frac{\Delta+2}{D}\partial\alpha\Phi^2 + \alpha\partial\Phi^2 + \frac{4\beta\Delta(\Delta+1)}{\gamma D^2(2\Delta+2-D)}\partial^2\partial\alpha\Phi^1$$
$$+\frac{4\beta(\Delta+1)}{\gamma D(2\Delta+2-D)}\partial_\mu\omega_{\mu\nu}\partial_\nu\Phi^1 + \frac{2\beta}{\gamma(2\Delta+2-D)}\omega_{\mu\nu}\partial_\mu\partial_\nu\Phi^1$$
$$+\frac{1}{\gamma D}\omega_{\mu\nu}\Phi^1_{\mu\nu} \tag{35}$$
$$\delta_\alpha \Phi^1_{\mu\nu} = \frac{\Delta}{D}\partial\alpha\Phi^1_{\mu\nu} + \alpha\partial\Phi^1_{\mu\nu} + \partial_\mu\alpha_\lambda\Phi^1_{\lambda\nu} + \partial_\nu\alpha_\lambda\Phi^1_{\mu\lambda} - \frac{\delta_{\mu\nu}}{D}\omega_{\lambda\chi}\Phi^1_{\lambda\chi}$$
$$-\frac{2\beta}{2\Delta+2-D}\omega_{\mu\nu}\partial_\lambda\partial_\lambda\Phi^1 + \frac{2\beta\Delta}{D}\left(\partial_\mu\partial_\nu\partial\alpha - \frac{\delta_{\mu\nu}}{D}\partial^2\partial\alpha\right)\Phi^1$$
$$+\beta\left(\partial_\mu\omega_{\lambda\nu} + \partial_\nu\omega_{\mu\lambda} - \frac{2}{D}\delta_{\mu\nu}\partial_\chi\omega_{\lambda\chi} - \partial_\lambda\omega_{\mu\nu}\right)\partial_\lambda\Phi^1 + \gamma\omega_{\mu\nu}\Phi^2.$$

Here,

$$\omega_{\mu\nu} \equiv \partial_\mu\alpha_\nu + \partial_\nu\alpha_\mu - \frac{2}{D}\delta_{\mu\nu}\partial\alpha.$$

In the D-dimensional theory of free scalar field ϕ, the operators

$$\Phi^1 \equiv \phi^2 \qquad \Phi^2 \equiv \phi\partial^2\phi$$
$$\Phi^1_{\mu\nu} \equiv \phi\partial_\mu\partial_\nu\phi - \frac{D}{4(D-1)}(\partial_\mu\partial_\nu)\phi^2 - \frac{\delta_{\mu\nu}}{D}\left(\phi\partial^2\phi - \frac{D}{4(D-1)}\partial^2\phi^2\right)$$

are transformed according to (35) with

$$\Delta = D-2, \beta = \frac{D-2}{8(D-1)} \qquad \gamma = \frac{1}{D}.$$

Thus, the family of the energy-momentum tensor $\Phi^1_{\mu\nu}$ in the free field theory contains two scalar fields Φ^1 and Φ^2, dimensions of which are D and $D-2$. One can suppose that this is right for non-trivial models, too and (35) describes by $\Delta = D-2$ the transformation law of these fields. For arbitrary D, there are arbitrary parameters β, γ in (35). For $D=2$, all the terms in (35) containing ω vanish by local conformal transformations, the field Φ^1 becomes a constant and the common formula for the energy-momentum tensor transformation [1] is obtained with one arbitrary parameter β, which is proportional to the central charge of the Virasoro algebra.

7. Conclusion

Our consideration seems to show that the use of the diffeomorphism group as the symmetry group in the D-dimensional conformal quantum field theory can be helpful for its investigation. The restrictions on the field operators obtained from this symmetry group enable us to extract many important informations about the system. From this point of view, the main problem is to study the special representations L_α of the diffeomorphism algebra, which were discussed above.

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References

[1] Belavin A A, Polyakov A M and Zamoldochikov A B 1984 *Nucl. Phys.* B **241** 333
[2] Mack G 1987 Introduction to conformal invariant quantum field theory in two and more dimensions, *Cargése Lectures* p 353
[3] Wilson K 1969 *Phys. Rev.* **179** 1499
[4] Ferrara S and Parisi G 1972 *Nucl. Phys.* **B42** 281
Ferrara S, Grillo A F and Gatto R 1973 *Ann. Phys., NY* **76** 161
Mack G 1977 *Commun. Math. Phys.* **53** 155
[5] Polyakov A M 1969 *Zh. Eksp. Teor. Fiz.* **28** 553
Migdal A A 1969 *Zh. Eksp. Teor. Fiz.* **28** 1036
[6] Fradkin E S and Palchik M Ya 1978 *Phys. Rep.* **44** 249
[7] Fradkin E S and Palchik M Ya 1990 *Preprint HU-TFT-90-11* University of Helsinki